INTERPHASE BOUNDARY DYNAMICS UNDER THE ACTION OF CAPILLARY FORCES. QUASISTEADY-STATE PLANE-PARALLEL MOTION

L. K. Antanovskii

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Slow plane-parallel motion of a viscous incompressible liquid with interphase boundary γ is considered in the absence of external and inertial mass forces. This implies that the complex velocity $v = v_x + iv_y$ and the pressure p satisfy a steady-state homogeneous Stokes system in the plane $z = x + iy$, with the change in the stress vector on the interphase boundary equal to the capillary forces, while the continuous velocity field defines the displacement of γ everywhere (phase transitions are absent). The proposed approximation is a natural one and can be justified at low Reynolds number and finite Struchal number.

For simplicity we assume that the surface tension coefficient σ is a known function of the point z and time t. For example, in the problem of thermocapillary convection this is true if the liquid temperature is known. It develops that in such a case for a fixed γ curve the dynamic condition defines the normal velocity component on γ in the form of an explicit operator $N(\gamma)$. We will call this operator the "normal velocity." Then the kinematic condition leads to the dynamic system

$$\dot{\gamma} = N(\gamma) \tag{0.1}$$

(where $\dot{\gamma}$ is the rate of motion of γ along its normal).

1. Fundamental Concepts. We introduce the notation of the Cauchy–Riemann operator $2\partial = \partial_x - i\partial_y$ and the external "stress" form $P(dz) = i(pdz + 2\mu\bar{\partial}vd\bar{z})$ (where ∂_x, ∂_y are operators representing partial differentiation with respect to x, y; μ is the dynamic viscosity coefficient, which will be considered piecewise-continuous with a discontinuity line γ). The differentials here are calculated at a fixed time t, and the explicit dependence of all quantities on t will not be indicated.

Let $z = \tau(s)$ be the parametrization of the curve γ with arc length s, $\nu = id\tau/ds$ being its normal. For the piecewise-continuous function f(z) with discontinuity line γ we take $f_{\pm}(\tau) = \lim\limits_{\varepsilon\to\pm 0} f(\tau + \varepsilon\nu), \tau \in \gamma,$ then write the fundamental problem in the form

$$dP(dz) = 0,\ d\,\mathrm{Im}(vd\bar{z}) = 0 \text{ outside } \gamma; \tag{1.1}$$

$$P_+(d\tau) - P_-(d\tau) = d(\sigma d\tau/ds),\ \dot{\gamma} = \mathrm{Re}(\bar{\nu}v) \text{ on } \gamma. \tag{1.2}$$

The velocity field will then be assumed continuous over the entire plane and vanishing at infinity, together with the pressure.

From Eq. (1.1) there follows the Kolosov–Muskhelishvili representation $v(z) = \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$, $iP(dz) = 2\mu d\,[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]$, where the functions $\varphi(z)$, $\psi(z)$ analytic outside γ are everywhere well defined. The latter follows from the easily proved identity $\int\limits_{\partial D} P\,(dz) = 0,$ where the region D contains the contour γ, consisting by definition of simple closed curves γ_j.

Let $\varphi_+(\tau) - \varphi_-(\tau) = \varphi\,(\tau)$; then from the continuity of the velocity there follows the equality $\psi_+(\tau) - \psi_-(\tau) = \overline{\omega\,(\tau)} - \bar{\tau}d\omega\,(\tau)/d\tau$. Since from the solution for v, p of the Stokes system the function φ is defined to the accuracy of the piecewise-continuous function $v(\infty) = 0$, it can be assumed that $\varphi(\infty) = \psi(\infty) = 0$. The additive constants in the function ω on the γ_j curves will be fixed below by interpretation of the dynamic condition [see Eq. (1.3)].

From the known change of the functions φ and ψ we construct Cauchy-type integrals, which lead to the basic equations $iP(dz) = 2\mu dU_\gamma\omega$, $v = V_\gamma\omega$, $\mathrm{Im}(vd\bar{z}) = dW_\gamma\omega$. Here U_γ, V_γ, W_γ are

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integral operators over ω, defined by the expressions $U_\gamma(\omega\,|\,z) = \frac{1}{2\pi i}\int\limits_\gamma \left[\omega(\tau)\, d\ln\left(\frac{\tau - z}{\bar\tau - \bar z}\right) - \overline{\omega(\tau)} \times d\left(\frac{\tau - z}{\bar\tau - \bar z}\right)\right]$, $V_\gamma(\omega\,|\,z) = \frac{1}{2\pi i}\int\limits_\gamma \left[\omega(\tau)\, d\ln|\tau - z|^2 + \overline{\omega(\tau)}\, d\left(\frac{\tau - z}{\bar\tau - \bar z}\right)\right]$, $W_\gamma(\omega\,|\,z) = \mathrm{Re}\,\frac{1}{2\pi}\int\limits_\gamma \overline{\omega(\tau)}\, d\,[(\tau - z)\ln|\tau - z|^2]$. It is obvious that the velocity $V_\gamma\omega$ and the flow function $W_\gamma\omega$ are continuous everywhere, while $(U_\gamma\omega)_\pm = \pm\omega + \widehat{U}_\gamma\omega$ ($\widehat{U}_\gamma$ being the main value of U_γ on γ). Integrating the dynamic condition, we obtain a Fredholm integral equation on the plane ω (an analog of the Sherman–Laurichell equation)

$$4\widehat{\mu}(\omega + \lambda\widehat{U}_\gamma\omega) = \sigma v,\ \widehat{\mu} = (1/2)(\mu_+ + \mu_-),$$
$$\lambda = (\mu_+ - \mu_-)/(\mu_+ + \mu_-), \tag{1.3}$$

where the integration constants on γ_j are dropped, which eliminates the arbitrariness in choice of ω. It can easily be seen that $\widehat{U}_\gamma c = \pm c$, $V_\gamma c = 0$, $W_\gamma c = 0$ (where c is a number constant on each γ_j, the sign of which is chosen depending on the orientation of γ_j).

We will demonstrate that Eq. (1.3) is always soluble, if $0 < \mu < \infty$ and γ is a Lyapunov curve. In the given case $|\lambda| < 1$ and the operator $\hat{U}_\gamma$ has a weak singularity, so that for proof of the solubility of Eq. (1.3) it is sufficient to establish the uniqueness of the zeroth solution at $\sigma = 0$. Using the integral identity $4\int\int \mu\,|\bar\partial v|^2\,dxdy + \mathrm{Re}\int\limits_\gamma \sigma\frac{dv}{ds}\,d\bar\tau = 0$ and the condition at infinity, we obtain an identically vanishing solution $v = 0$, $p = 0$.

Consequently, the function φ is piecewise-continuous as is ω, so that homogeneous equation (1.3) leads to the equality $\omega \pm \lambda\omega = 0$ or $\omega = 0$. Thus the solution of Eq. (1.3) can be represented in the form $4\hat{\mu}\omega = \sigma v - \lambda R_\lambda(\sigma v)$ (where R_λ is an integral operator).

2. Realization of the "Normal Velocity" Operator. The operator $N(\gamma)$ is now completely concretized: $N(\gamma) = dF(\tau)/ds$, $\tau \in \gamma$. Here $4\widehat{\mu}F(\tau) = W_\gamma(\sigma v - \lambda R_\lambda(\sigma v))$ or in expanded form $4\widehat{\mu}F(\tau_0) = \frac{1}{2\pi}\int\limits_\gamma \mathrm{Im}\left(\frac{d\bar\tau}{d\tau}\frac{\tau - \tau_0}{\bar\tau - \bar\tau_0}\right)\sigma(\tau)\,ds - \mathrm{Re}\,\frac{\lambda}{2\pi}\int\limits_\gamma \overline{R_\lambda(\sigma v\,|\,\tau)}\,d\,[(\tau - \tau_0)\ln|\tau - \tau_0|^2]$. Let a dot above denote differentiation with respect to time; then system (0.1) transforms to the equation

$$\mathrm{Im}(\dot\tau d\bar\tau) = dF(\tau), \tag{2.1}$$

because $\dot\gamma = \mathrm{Im}(\dot\tau d\bar\tau/ds)$. We note that now in Eq. (2.1) the position of the point τ on γ can be specified by any parameter.

We assume that γ is a counterclockwise curve, bounding a finite region (planar "droplet"). Let πr^2 be its area, and a its center of mass, i.e., $2\pi r^2 = \mathrm{Im}\int\limits_\gamma \bar\tau d\tau$, $a = \frac{1}{2\pi i r^2}\int\limits_\gamma |\tau|^2\,d\tau$. It follows from Eq. (2.1) that the area of the "droplet" is preserved, while the point a moves according to a law

$$\dot a = \frac{1}{\pi r^2}\int\limits_\gamma F(\tau)\,d\tau. \tag{2.2}$$

We will consider the manifold M of real 2π-periodic continuous functions $\eta(\theta)$ satisfying the conditions $\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{2\eta}d\theta = 1$, $\int\limits_{-\pi}^{\pi} e^{3\eta + i\theta}d\theta = 0$. Then the class of curves stellate relative to the point a can be specified by the element $\eta \in M$ in the form $\gamma = \{\tau = a + re^{\eta + i\theta}\}$. As a result Eq. (2.1) transforms to a system for a and η:

$$\dot a = A(a, \eta); \tag{2.3}$$

$$(e^{2\eta}/2)^{\cdot} + B'(\eta, a) = 0, \tag{2.4}$$

where $A(a, \eta)$ is the right side of Eq. (2.2); $B(\eta, a|\theta) = r^{-2}\{\mathrm{Im}[\dot a(\bar\tau - \bar a)] - F(\tau)\}$. Here and below the prime denotes differentiation with respect to θ. It is obvious that an analogous construction can be performed for a set of closed curves.

3. Exact Solutions. Since B(0, a) = 0 for an arbitrary function $\sigma(z)$, then within the class of circles there exists a solution of system (0.1), if their centers satisfy the ordinary differential equation

$$\dot{a} = A(a, 0) \equiv -\frac{1}{8\pi\mu}\int_{-\pi}^{\pi} \sigma(a + re^{i\theta})\, e^{i\theta} d\theta. \tag{3.1}$$

We introduce the positive function

$$\alpha(\theta) = \sigma(a + re^{i\theta})/4\widehat{\mu}r \tag{3.2}$$

and the Hilbert operator H with the expression $H(f\,|\,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\operatorname{ctg}\frac{\theta-\xi}{2}d\xi$. Then the liquid velocity on γ is defined in the form $v(a + re^{i\theta}) = \dot{a} - rie^{i\theta}H(\alpha|\theta)$. For example, if $\sigma(z) = \sigma_0[1 + \varepsilon(|z/r|^2 - 1)]$, then by choosing as the time scale $4\widehat{\mu}r/\sigma_0$, we obtain the solution $a = a_0e^{-\varepsilon t}$, $v(a + re^{i\theta}) = -\varepsilon\bar{a}e^{2i\theta}$ (where a_0 is the center of the initial circle and r is its radius).

Linearization of Eq. (2.4) at the equilibrium position $\eta = 0$, $a = 0$ leads to the problem $\dot{\eta} + (H\eta)' = 0$ for the function η orthogonal to 1 and $e^{i\theta}$. If $\eta_0(\theta)$ is the initial perturbation of γ, then $\eta(\theta) = \frac{e^{-2t}}{\pi}\int_{-\pi}^{\pi}\frac{\cos 2(\theta-\xi) - e^{-t}\cos(\theta-\xi)}{1 - 2e^{-t}\cos(\theta-\xi) + e^{-2t}}\eta_0(\xi)\,d\xi$. It is interesting that with increase in time $a \to 0$ for $\varepsilon > 0$, while $\eta \to 0$ for all ε. Our goal is to prove this last assertion in more general form.

4. Stability of the Interphase Boundary Form. Now for simplicity let $\lambda = 0$, so that the problem of Eq. (2.4) linearized to the exact solution $\eta = 0$ has the simple form

$$\dot{\eta} + [H(\alpha\eta) - \eta H\alpha - K(\alpha, \eta)]' = 0, \tag{4.1}$$

where the coefficient α is defined by Eq. (3.2) and $K(\alpha, \eta\,|\,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi}\eta(\xi)[\alpha(\xi)\sin(\theta-\xi) - H(\alpha|\xi)\cos(\theta-\xi)]d\xi$. It is obvious that the set T_0M of functions η orthogonal to 1 and $e^{i\theta}$ (tangent space at the null of the manifold M) is invariant with respect to the operator of Eq. (4.1). We introduce in the set T_0M the structure of a Sobolev space with norm $\|\eta\|_s = \left(\sum_{|k|\geqslant 2}|k|^{2s}|\eta_k|^2\right)^{1/2}$, $\eta_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\eta(\theta)e^{-ik\theta}d\theta$ and define the two following numbers: $\alpha_* = \inf_{\theta,t}\alpha$, $\delta = \inf_t \alpha_0 - 2\sup_t\sum_{|k|\neq 0}|\alpha_k|$ (it being obvious that $\delta \leqslant \alpha_*$). Then the estimates

$$\|\eta\|_0 \leqslant \|\eta_0\|_0 e^{-2\alpha_* t}, \quad \|\eta\|_s \leqslant \|\eta_0\|_s e^{-2\delta t}, \quad s \geqslant 0 \tag{4.2}$$

are valid (where η_0 is the initial perturbation). In particular, for $\delta > 0$ and $s > 1/2$ we have asymptotic stability of the zeroth solution at the norm of the continuous functions.

A first estimate is easily obtained by multiplication of Eq. (4.1) by η and integration over period, which leads to the identity $(\|\eta\|_0^2/2)^{\cdot} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\alpha(\xi)\left(\frac{\eta(\theta)-\eta(\xi)}{2\sin\frac{\theta-\xi}{2}}\right)^2 d\xi d\theta = 0$. Since the second term for $\alpha = 1$ coincides with $\|\eta\|_{1/2}^2$, by Gronwall's lemma the first assertion is valid. The second inequality is based on evaluation of the commutator $\|H(\alpha\eta) - \eta H\alpha\|_s \leqslant 2\Sigma|\alpha_k|\,\|\eta\|_s$, $s \geqslant 0$. In fact,

$$\|H(\alpha\eta) - \eta H\alpha\|_s^2 = \sum_k\left||k|^s\sum_l(\operatorname{sgn}k - \operatorname{sgn}l)\eta_{k-l}\alpha_l\right|^2 \leqslant$$

$$\leqslant \sum_k\left(2\sum_l|k-l|^s|\eta_{k-l}||\alpha_l|\right)^2 \leqslant 4\sum_k\sum_l|k-l|^{2s}|\eta_{k-l}|^2|\alpha_l|\sum_m|\alpha_m| = \left(2\sum|\alpha_k|\right)^2\|\eta\|_s^2.$$

Multiplying Eq. (4.1) scalarly by the function $\sum_k|k|^{2s}\eta_k e^{ik\theta}$, we obtain the inequality $(\|\eta\|_s^2/2)^{\cdot} + \alpha_0\|\eta\|_{s+1/2}^2 \leqslant 2\sum_{k\neq 0}|\alpha_k|\,\|\eta\|_{s+1/2}^2$, from which Eq. (4.2) follows. In applying Gronwall's lemma it is necessary to use the estimate $\|\eta\|_{s+1/2} \geqslant \sqrt{2}\|\eta\|_s$, valid for $\eta \in T_0M$.

The stability condition $\delta > 0$ can be ensured by requiring that the analog of the Marangoni number $\mathrm{Ma} = r \sup|\partial\sigma|/\inf\sigma$ lie in the interval [0, Ma*]. From general considerations Ma defines the ratio of the velocity of drift of the center of mass a [see Eq. (3.1)] to the velocity of "rounding" of γ, while since in real situations Ma is limitingly small, it is obvious that the dynamics of the interphase boundary are close to those of a circle, which is in fact described by one equation of Eq. (3.1).

Notes. 1. The Kolosov–Muskhelishvili representations and Sherman–Laurichell equations [1] can be extended from elasticity theory almost without change to viscous liquid hydrodynamics (see [2, 3] and literature cited therein). The methods of the theory of functions of a complex variable were first applied to steady-state problems with a free boundary for the Navier–Stokes equation by the author of [4-8]. The exact solutions found herein were in fact obtained in [5], while [6, 7] established the isolated nature of some of these.

2. To prove Eq. (4.2) the author used an inequality for the commutator [9]. It is interesting that the linearized equation (4.1) contains no arbitrary surface tension coefficient, while on the other hand, α is smoothed by the commutator. Thus Eq. (4.1) is an example of an equation with a quite irregular coefficient (the absolutely converging Fourier series defines a continuous, but generally speaking, nondifferentiable function), but having a solution as smooth as desired if the initial conditions are smooth.

3. The operator N can also be realized in the spatial case. The velocity field must be represented by the potential of a simple layer [3], and an analog of Eq. (1.3) is obtained, from which the vector density is uniquely determined, after which dynamic system (0.1) is written. It proves to be the case that many properties of the "normal velocity" operator are preserved. Thus, the planar formulation of the problem can be considered a Lotsman model.

4. For constant σ the problem of evolution of a finite liquid volume was studied in precise formulation in [10] (see literature cited therein). For low Marangoni number, [11] found an asymptotically exact solution of the problem of thermocapillary propulsion of a spherical droplet of viscous liquid within another liquid.

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